RESEARCH STATEMENT: NUMBER THEORY IN FUNCTION FIELDS.

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1. Scientific Overview

My research focuses on the study several analogues of classical analytic number theoretical problems in the context of function fields over a large finite field.

There are remarkable similarities between the ring of integers $\mathbb{Z}$ and the polynomial ring $\mathbb{F}_q[t]$ over the finite field $\mathbb{F}_q$. In at least one respect, it is surprising that these rings resemble one another, as the characteristic of $\mathbb{Z}$ is zero, whereas that of $\mathbb{F}_q[t]$ is positive. A significant necessity in translating results from $\mathbb{Z}$ to $\mathbb{F}_q[t]$, therefore, is the derivation of methods independent of the characteristic.

This observation lead in recent years to flourishing research on number theory in function fields, with many astonishing achievements. For instance, Katz-Sarnak's [29] result on equidistribution of Frobenius conjugacy classes, Keating-Rudnick's calculation of the variance of primes in short intervals (Hooley's conjecture) and in arithmetic progressions (Goldston-Montgomery's conjecture) in [30], and the works of Liu and Wooley on the Waring problem [37, 38]. Several more works are those of Entin [18] on the Bateman-Horn conjecture, Rodgers [46, 45] on almost primes and on arithmetic functions, Bender and Pollack [12, 13] on the quantitative polynomial Goldbach's problem, and the Hardy-Littlewood (HL) prime polynomial n-tuple problem which was resolved by Bary-Soroker in [10] in odd characteristic and by Carmon in [15].

These problems in analytic number theory that were addressed in the context of function fields require completely different methods than those of traditional analytic number theory. Their resolutions in the function field setting is encouraging, as most experts consider some of these problems intractable in the classical setting.

One similarity between $\mathbb{Z}$ and $\mathbb{F}_q[t]$ that I study is the density of the prime elements in the mentioned rings in short intervals. When considering the prime elements as the building blocks of $\mathbb{Z}$ or $\mathbb{F}_q[t]$, there is no apparent reason why the "same" density of primes should hold in both rings. However, the Prime Number Theorem (PNT) and its function field analogue, the Prime Polynomial Theorem (PPT) state that the density of prime integers up to a large x and the density of prime polynomials of a given degree are the same if formulated in the correct language.

In [7], we used a new approach in solving the question of primes polynomials in short intervals and in arithmetic progressions with large modulus, both still open questions in the setting of integers. This approach involves computations of Galois groups as well as using explicit Chebotarev density theorems. The new approach enables us to calculate asymptotic formulas that go much further than what the Riemann Hypothesis implies.

Recalling that the ring $\mathbb{F}_q[t]$ can be viewed as the ring of rational functions on $\mathbb{P}^1$, regular away from infinity, one may further wonder whether the density of primes remains the same

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when considering primes in a ring of rational functions of any smooth projective curve C defined over a finite field $\mathbb{F}_q$, regular away from some divisor E on C. In [8], we extend the definition of a short interval to any curve C as above, and show that indeed the expected density of primes still holds. The extension of the definition of a short interval, as well as the computation of the density of primes, required an algebro-geometric point of view. Detailed descriptions of these two results are briefly described in Sections 2.2 and 2.3, and fully described in Section 3.1 and Section 3.2.

As for correlations between primes, namely the HL conjecture and Goldbach's conjecture, it turns out that the approach of exploiting Galois theory together with Chebotarev-type density theorems works well. The additional elements that are needed here are several field arithmetic arguments and the study of the discriminants of certain field extensions. In [5], we calculate an asymptotic formula for the number of simultaneous prime polynomial values of several linear functions in short intervals. This asymptotic formula generalizes some of the above problems and moreover, it resolves these problems in the setting of short intervals. In [4], we continue the study of correlations between primes in short intervals in the setting of smooth projective curves defined over finite fields. Our result gives an asymptotic formula for the desired density. We briefly review these in Sections 2.4 and 2.5, and at length in Section 3.3 and Section 3.4.

A question similar in nature arises from Landau's theorem on the density of integers which are sums of two square integers. In [6], we formulate a function field analogue of this theorem in short intervals, and show that the desired density still holds. A short description of this project is given in Section 2.6, and a detailed one in Section 3.5.

Another similarity between $\mathbb{Z}$ and $\mathbb{F}_q[t]$ appears when considering problems in Diophantine approximations. Here, as before, there is a straightforward translation between the two settings. Approximating a real number by an integer translates to approximating a Laurent polynomial in $\mathbb{F}_q((\frac{1}{t}))$ by a polynomial in $\mathbb{F}_q[t]$. In [9], we address a problem in inhomogeneous Diophantine approximation in the function field setting, to attain Hausdorff dimension results and to give an explicit Cassels' constant to this problem. This work is described briefly in Section 2.7, and at length in Section 3.6.

2. Short description of my work

2.1. Integers vs Polynomial rings. Before we state our previous work, it might be useful to briefly recall some of the analogies between the integers and the polynomial ring $\mathbb{F}_q[t]$ that are used throughout. We summarize these below.

(1)

$\mathbb{Z}$	ring of polynomials $\mathbb{F}_q[t]$
$\lvert x\rvert$	$\lvert f\rvert \overset{\text{def}}{=} q^{\deg f}$
$(0,x]$	$M(k,q) \overset{\text{def}}{=} \{h \in \mathbb{F}_q[t] : h \text{ is monic and } \deg h = k\}$
$x = \#(0,x]$	$q^k = \#M(k,q)$
$\log x$	$k = \log_q q^k$
prime number	prime polynomial $\overset{\text{def}}{=}$ monic and irreducible polynomial

2.2. Primes in short intervals and in arithmetic progressions. In [7], we used a new approach in solving the questions of primes polynomials in short intervals and in arithmetic progressions with large modulus. Both of these are still open questions in the classical setting. This approach involves computations of Galois groups as well as using explicit Chebotarev density theorems. Explicitly, in [21, p. 7] Granville conjectures

Conjecture 2.1. For a fixed ε with $0 < \varepsilon < 1$

$$\pi(I(x,\varepsilon)) \sim \frac{\#I(x,\varepsilon)}{\log x}. \tag{2}$$

Here, π is the prime counting function and $I(x,\varepsilon) \stackrel{\text{def}}{=} (x - x^{\varepsilon}, x + x^{\varepsilon})$ is a *short interval.* Heath-Brown [25], improving Huxley [28], proves Conjecture 2.1 unconditionally for $\frac{7}{12} < \varepsilon$. However, the conjecture in full generality is still open. Using Table (1), a *short interval* $I = I(f_0,\varepsilon)$ around a polynomial $f_0 \in \mathbb{F}_q[t]$ with $0 < \varepsilon < 1$ fixed, is defined as

$$I(f_0,\varepsilon) \stackrel{\text{def}}{=} \{f \in \mathbb{F}_q[t] : |f - f_0| \leq |f_0|^{\varepsilon}\} = f_0 + \mathcal{P}_{\leq \lfloor \varepsilon \deg f_0 \rfloor}. \tag{3}$$

For the prime polynomial counting function $\pi_q(I(f_0,\varepsilon))$, we prove

Theorem 2.1. [7, Theorem 2.3] Let k be a positive integer and $\frac{3}{k} \leq \varepsilon < 1$. Then the asymptotic formula

$$\pi_q(I(f_0,\varepsilon)) = \frac{\#I(f_0,\varepsilon)}{k}\left(1 + O_k(q^{-1/2})\right) \tag{4}$$

holds uniformly for all prime powers q, monic polynomials $f_0 \in \mathbb{F}_q[t]$ of degree k and short intervals $I(f_0,\varepsilon)$.

Remark 2.2. Let us mention that in [7] we prove much more;

- We fully characterize the cases where Theorem 2.1 holds or fails for $0 < \varepsilon < \frac{3}{k}$.
- We prove a result on the density of primes in an arithmetic progressions with large modulus [7, Theorem 2.5].
- We establish results that deal with general factorization types [7, Proposition 3.1].

In order to make this document short and clear, we chose not to bring the additional results.

2.3. **Primes in short intervals on curves of higher genus.** From a geometric point of view, polynomials are effective zero-cycles on the affine line. This leads one to ask: Can the analogy between integers and polynomials be extended to zero-cycles on more general varieties? In [8], we consider smooth projective geometrically irreducible curves of arbitrary genus defined over a finite field. We introduce a generalized definition of a short interval on curves, and show that the expected density of primes in such intervals hold. More precisely, we define

Definition 2.2. Let $E = m_1\mathfrak{p}_1 + \cdots + m_s\mathfrak{p}_s$ be an effective divisor on C, and let f_0 be a regular function on the complement of E. The *interval* (*of size E around f_0*) is the set

$$\begin{aligned} I(f_0,E) &\stackrel{\text{def}}{=} \left\{ \begin{matrix} \text{regular functions } h \text{ on } C\backslash \mathrm{supp}(E) \text{ such} \\ \text{that } \nu_{\mathfrak{p}_i}(h - f_0) \geq -m_i \ \text{ for all } \ 1 \leq i \leq s \end{matrix} \right\} \\ &= f_0 + H^0\big(C, \mathscr{O}(E)\big), \end{aligned} \tag{5}$$

The interval $I(f_0,E)$ is a *short interval* if the order of the pole of f_0 at each $\mathfrak{p}_i$ is strictly greater than m_i.

We prove

Theorem 2.3. [8, Theorem A.] Let C be a curve of genus g over $\mathbb{F}_q$ as above, and let $k > 0$ be an integer. Let $I(f_0,E)$ be a short interval. Assume in addition that E is sufficiently positive. Then

$$\pi_C\big(I(f_0,E)\big) = \frac{\#I(f_0,E)}{k}\Big(1 + O_k(q^{-1/2})\Big). \tag{6}$$

Furthermore, the asymptotic formula (6) is uniform in E and f_0. Here, $k \stackrel{\text{def}}{=}$ total order of poles of f_0 on C and $\pi_C\big(I(f_0,E)\big)$ is the principal prime ideal counting function.

Remark 2.3. We note that

- The conditions on E being sufficiently positive are fully explained in [8].
- We present analogous conjectures to the Theorem 2.3 for general number fields [8, Section 1.3].
- We prove a stronger result that deal with other factorization types [8, Proposition 5.1.4].

2.4. **Prime polynomial values of linear functions in short intervals.** The quantitative Goldbach conjecture and the Hardy-Littlewood (HL) n-tuple conjecture are long standing, extensively studied, unresolved conjectures in number theory. A crucial observation is that the two problems are specific cases of counting the number of simultaneous prime values of linear functions. In other words, consider n distinct primitive functions $L_i(X) = a_i + b_i X$ with $a_i, b_i \in \mathbb{Z}$. Write $\mathcal{L} = (L_1, ..., L_n)$ and let $\pi_{\mathcal{L}}(I(x, \varepsilon))$ be the corresponding prime counting function in the interval $I(x, \varepsilon)$. As in the heuristic derivation of the HL conjecture from the PNT, one expects

Conjecture 2.4.

$$\pi_{\mathcal{L}}(I(x,\varepsilon)) \sim \mathfrak{S}(L_1, \ldots, L_n) \frac{\#I(x,\varepsilon)}{\prod_{i=1}^{n} \log(L_i(x))} \qquad \text{as} \quad x \to \infty \tag{7}$$

where $\mathfrak{S}(L_1, \ldots, L_n)$ is a positive constant, and under some restrictions on a_i, b_i.

Note that if $L_1(x) = x$ and $L_2(x) = a - x$, then Conjecture 2.4 imply a quantitative Goldbach conjecture, and if $L_i(X) = x + a_i$ retrieve the HL n-tuple conjecture, both in short intervals. The Polynomial Goldbach problem and the HL problem were both settled by Pollack and Bender [13] and independently by Bary-Soroker [10], all in the large q limit. In [5], we prove an analogue of Conjecture 2.4; For a primitive linear function $L(X) = f(t) + g(t)X$ with $f, g \in \mathbb{F}_q[t]$ and $g \neq 0$, the *height of* L is $\mathrm{h}(L) \stackrel{\text{def}}{=} \max\{\deg(f), \deg(g)\}$. Let $\pi_{q,\mathcal{L}}(I(f_0, \varepsilon))$ be the corresponding prime counting function. We prove

Theorem 2.4. [5, Theorem 1.1] Let $0 < B$ and $0 < \varepsilon < 1$ be fixed real numbers. Then the asymptotic formula

$$\pi_{q,\mathcal{L}}(I(f_0,\varepsilon)) = \frac{\#I(f_0,\varepsilon)}{\prod_{i=1}^{n} \deg(L_i(f_0))} \Big(1 + O_B(q^{-1/2})\Big)$$

holds uniformly for all odd prime powers q, $1 \leq n \leq B$, distinct primitive linear functions $L_1(X), \ldots, L_n(X)$ defined over $\mathbb{F}_q[t]$ each of height at most B, and monic $f_0 \in \mathbb{F}_q[t]$ of degree in the interval $\frac{2}{\varepsilon} \leq \deg f_0 \leq B$.

2.5. **Correlation between primes on curves.** As in the Section 2.3, one may ask similar question about the correlation of primes in the setting of curves over a finite field. For a smooth projective geometrically irreducible curve C of genus g over a finite field $\mathbb{F}_q$, we let $E = m_1\mathfrak{p}_1 + \cdots + m_s\mathfrak{p}_s$ be an effective divisor on C. Let $\sigma \stackrel{\text{def}}{=} (\sigma_1, ..., \sigma_n)$ be an n-tuple of distinct rational functions on C, regular on $C \backslash \mathrm{supp}\,(E)$. In [4], we prove

Theorem 2.5. [4, Theorem A] Fix an integer $B > n$. If $\mathrm{char}\, \mathbb{F}_q \neq 2$ and $E \geq 3E_0$ for some effective divisor E_0 on C with $\deg E_0 \geq 2g + 1$, then the asymptotic formula

$$\pi_{C,\sigma}\big(I(f_0, E)\big) \;=\; \frac{\#I(f_0, E)}{\prod_{i=1}^{n} \deg \mathrm{div}(f_0 + \sigma_i)|_{C \backslash E}} \Big(1 + O_B\big(q^{-1/2}\big)\Big)$$

holds uniformly for all E and $f_0, \sigma_1, \ldots, \sigma_n$ as above satisfying $\deg\big(\mathrm{div}(f_0 + \sigma_i)\big|_E\big) < B$, and as $q \to \infty$ an odd prime power. Here, $\pi_{C,\sigma}\big(I(f_0, E)\big)$ is the natural principal prime counting function.

2.6. **Sum of squares in short intervals.** In this project we prove an asymptotic density for the number of polynomials that can be represented as a sum of two squares in short intervals. An integer $n \in \mathbb{Z}$ is *representable as a sum of two squares*, in short *representable* if there are $a, b \in \mathbb{Z}$ such that $n = a^2 + b^2$. Equivalently, an integer is representable if and only if it is a norm of an element of the Gaussian integers $\mathbb{Z}[i]$. A famous theorem of Landau [36], asserts that the density of representable integers up to x is $\frac{K}{\sqrt{\log x}}$, where K is the Landau-Ramanujan constant. The problem of estimating this density in intervals $I(x, \Phi(x)) \stackrel{\text{def}}{=} (x - \Phi(x), x + \Phi(x))$ has a long history. There are many results [20, 19, 26, 23, 43] giving correct upper and lower bounds in an almost everywhere sense. However, for $\Phi(x) \sim (\log x)^A$, $A > \frac{1}{2}$, Balog and Wooley [3] show that the expected density fails. Using methods of Ingham, Montgomery and Huxley it can be confirmed that the density of representable elements in an interval $I(x, \varepsilon)$ with $\varepsilon > \frac{7}{12}$ is $\#I(x, \varepsilon) \cdot \frac{K}{\sqrt{\log x}}$ as expected (see [27]). In [6], we resolve a function field analogue of the above. Consider the ring $\mathbb{F}_q[\sqrt{-t}]$ as analogous of $\mathbb{Z}[i]$. A polynomial $f \in \mathbb{F}_q[t]$ is *representable* if it is a norm of an element of $\mathbb{F}_q[\sqrt{-t}]$. We extend the work in [11], and prove

Theorem 2.6. [6, Theorem] For odd q, $k > 2$, $1 > \varepsilon \geq \frac{2}{k}$, and $f_0 \in \mathbb{F}_q[t]$ monic of degree k

$$\#\{f \in I(f_0, \varepsilon) : f \text{ is representable } \} = \#I(f_0, \epsilon)\Big(\frac{1}{4^k}\binom{2k}{k} + O_k(q^{-1/2})\Big) \tag{8}$$

where the implied constant depends only on k. For $0 < \epsilon < \frac{2}{k}$, (28) no longer holds.

2.7. **Diophantine approximations in function fields.** A main topic in Diophantine approximation deals with the inhomogeneous approximation of a real number. In this project we prove a function field analogue of this problem. Function fields analogues of Diophantine approximation have been studied since the works of Artin [1] and Mahler [39]. Recently, this subject has regained interest, parallel to a significant progress in the real case, see for example [32, 33, 48]. In [9], we prove Hausdorff dimension results for exceptional sets connected to inhomogeneous Diophantine approximation and determine explicitly Cassels' constant, both in the function fields setting. In the (classical) real case, similar Hausdorff dimension results have also been proved. However, only estimates have been obtained for Cassels' constant. We prove

Theorem 2.7. [9, Theorem 3.5] For every $\theta \in \mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)$, $\dim(BA_\theta) = 1$.

Theorem 2.8. [9, Theorem 3.9] Cassels' constant is $c = q^{-2}$.

Remark 2.5. We remark that in [9] we prove more. In fact, we prove the above theorems in higher dimensions [Theorem] and for general weights [Theorem].

3. Previous Work

3.1. **Primes in short intervals and in arithmetic progressions.** One classical subject in number theory is counting the number of primes, for example in short intervals and in arithmetic progressions. Keeping in mind the Prime Number Theorem (PNT), one may expect that a real interval $I = I(x, \Phi(x)) = (x - \Phi(x), x + \Phi(x)]$ of size $2\Phi(x)$ starting at a large x contains about $2\Phi(x)/\log x$ primes. More precisely, if we let $\pi(x)$ be the prime counting function

(9)

$$\pi(x) \stackrel{\text{def}}{=} \#\{2 < p \leq x : p \text{ is a prime number}\};$$

$$\pi(I(x, \Phi(x))) \stackrel{\text{def}}{=} \#\{p \in I(x, \Phi(x)) : p \text{ is a prime number}\} = \pi(x + \Phi(x)) - \pi(x - \Phi(x)).$$

Then one expects that

$$\pi(I) \sim \int_I \frac{dt}{\log t} \sim \frac{|I|}{\log x}. \tag{10}$$

The PNT implies that (10) holds for $\Phi(x) \sim cx$, for any fixed $0 < c < 1$. Under the Riemann Hypothesis, (10) holds for $\Phi(x) \sim \sqrt{x}\log x$. Concerning smaller powers of x Granville conjectures [21, p. 7]

Conjecture 3.1. If $\Phi(x) > x^{\varepsilon}$ with $0 < \varepsilon < 1$ then (10) holds.

Heath-Brown [25], improving Huxley [28], proves Conjecture 3.1 unconditionally for $\frac{7}{12} < \varepsilon$. We note that for extremely short intervals, (10) fails uniformly [44], but may hold for almost all x, see [49] and the survey [22, Section 4]. The rest of this document therefore refers to an interval of the form $I = I(x,\varepsilon) = (x - x^{\varepsilon}, x + x^{\varepsilon}]$ as a *short interval*.

In a paper published in Duke [7], together with my collaborators, I prove an analogue of Conjecture 3.1 in function fields. Using Table (**??**), a *short interval* $I = I(f_0, \varepsilon)$ around a polynomial $f_0 \in \mathbb{F}_q[t]$ with $0 < \varepsilon < 1$ fixed, is defined as

$$I(f_0,\varepsilon) \stackrel{\text{def}}{=} \{f \in \mathbb{F}_q[t] : |f - f_0| \leq |f_0|^{\varepsilon}\} = f_0 + \mathcal{P}_{\leq \lfloor \varepsilon \deg f_0 \rfloor}. \tag{11}$$

Here, $\mathcal{P}_{\leq \lfloor \varepsilon \deg f_0 \rfloor}$ is the space of polynomials of degree up to $\lfloor \varepsilon \deg f_0 \rfloor$. The prime counting function is then

$$\begin{aligned} \pi_q(k) &\stackrel{\text{def}}{=} \#\{h \in \mathbb{F}_q[t] : h \text{ is a prime polynomial with } \deg h = k\}; \\ \pi_q(I(f_0,\varepsilon)) &\stackrel{\text{def}}{=} \#\{h \in I(f_0,\varepsilon) : h \text{ is a prime polynomial}\}. \end{aligned} \tag{12}$$

We prove

Theorem 3.1. [7, Theorem 2.3] Let k be a positive integer and $\frac{3}{k} \leq \varepsilon < 1$. Then the asymptotic formula

$$\pi_q(I(f_0,\varepsilon)) = \frac{\#I(f_0,\varepsilon)}{k}\left(1 + O_k(q^{-1/2})\right) \tag{13}$$

holds uniformly for all prime powers q, monic polynomials $f_0 \in \mathbb{F}_q[t]$ of degree k, and short intervals $I(f_0,\varepsilon)$.

Remark 3.2. Let us mention that in [7] we fully characterize the cases where Theorem 3.1 holds or fails for $0 < \varepsilon < 3/k$. In order to make this document short and clear, we chose not to bring these cases here.

For primes in arithmetic progressions with large modulus, we obtain a similar result

Theorem 3.2. [7, Theorem 2.5] Let k be a fixed integer and $\frac{3}{k} \leq \delta \leq 1$. Consider the counting function

$$\pi_q(k; D, f) \stackrel{\text{def}}{=} \{h \equiv f(\mod D) : h \text{ is prime, and } \deg h = k\}.$$

Then the asymptotic formula

$$\pi_q(k; D, f) = \frac{\pi_q(k)}{\phi(D)}\left(1 + O_k(q^{-1/2})\right) \tag{14}$$

holds uniformly for all prime powers q and relatively prime $f, D \in \mathbb{F}_q[t]$ satisfying $|D| \leq q^{\deg f(1-\delta_0)}$. Here, $\phi(D)$ is the Euler totient function.

Remark 3.3. We remark that in order to establish Theorems 3.1 and 3.2, we prove results which are stronger in the sense that they deal with general factorization types. Hence, they can be applied to other functions as well, like the ℓ-th divisor function.

3.2. **Primes in short intervals on curves of higher genus.** Recalling that the polynomial ring $\mathbb{F}_q[t]$ may be viewed as the ring of rational functions on $\mathbb{P}^1$ regular away from ∞, and in light of Theorem 3.1, a natural question asks whether similar estimates of the density of primes in short intervals hold for arbitrary curves over a finite field. From the perspective of curves over finite fields, the analogue of the PPT is (see [47, Theorem 5.12])

$$\pi_C(k) = \frac{q^k}{k}\left(1 + O(q^{-k/2})\right). \tag{15}$$

Where C is a smooth projective geometrically irreducible curve over $\mathbb{F}_q$, and the prime counting function is

$$\pi_C(k) = \#\{P \text{ a prime divisor of } C : \deg(P) = k\}.$$

Another perspective one might take comes from Landau's Prime Ideal Theorem (PIT) [35] and a Principal Prime Ideal Theorem (PPIT)[42, Section 7.2] for number fields. Since our work concerns mainly the principal case, we provide here only the formulation of PPIT. For K an algebraic number field of degree n over $\mathbb{Q}$ with class number h_K, we have

$$\begin{aligned} \pi_{K,\text{prin}}(x) &\overset{\text{def}}{=} \#\{\text{principal prime ideals } (a) \subset \mathcal{O}_K : \ 2 < N_K(a) \leq x\} \\ &\sim \tfrac{1}{h_K} \cdot \tfrac{x}{\log x} \quad \text{as} \quad x \to \infty. \end{aligned} \tag{16}$$

As for short intervals in a general number field K, one may conjecture two possible analogues of Conjecture 3.1

Conjecture 3.4. Let $S = \{\text{infinite places of } K\}$. There exists some constant c such that for each real vector $\varepsilon_S = (\varepsilon_\mathfrak{p})_{\mathfrak{p}\in S}$ in $(0,1)^S \subset \mathbb{R}^S$, the count

$$\pi_{K,\text{prin}}\big(I(b,\varepsilon_S)\big) = \#\big\{a \in \mathcal{O}_K : |a-b|_\mathfrak{p} \leq |b|_\mathfrak{p}^{\varepsilon_\mathfrak{p}} \text{ for each } \mathfrak{p} \in S, \text{ and } (a) \subset \mathcal{O}_K \text{ is prime}\big\}$$

satisfies the asymptotic formula

$$\pi_{K,\text{prin}}\big(I(b,\varepsilon_S)\big) \sim c \cdot \frac{\#\{a \in \mathcal{O}_K : |a-b|_\mathfrak{p} \leq |b|_\mathfrak{p}^{\varepsilon_\mathfrak{p}} \text{ for all } \mathfrak{p} \in S\}}{\log N_K(b)} \quad \text{as} \quad N_K(b) \to \infty. \tag{17}$$

Conjecture 3.5. There exists some constant c such that for each $0 < \varepsilon < 1$, the count

$$\pi_{K,\text{prin}}\big(I(x,\varepsilon)\big) = \#\big\{\text{principal prime ideals } (a) \subset \mathcal{O}_K : \ x - x^\varepsilon < N_K(a) \leq x + x^\varepsilon\big\}.$$

satisfies the asymptotic formula

$$\pi_{K,\text{prin}}\big(I(x,\varepsilon)\big) \sim c \cdot \frac{\#I(x,\varepsilon)}{\log x} = c \cdot \frac{2\,x^\varepsilon}{\log x} \quad \text{as} \quad x \to \infty. \tag{18}$$

Balog and Ono [2], using formulas for the prime ideal counting function due to Lagarias and Odlyzko [34] and zero density estimates for Dedekind zeta-functions due to Heath-Brown [24] and Mitsui [40], show that Conjecture (3.5) holds for $1 - \frac{1}{n} < \varepsilon \leq 1$. Here one may take $n = 8/3$ if $[K : \mathbb{Q}] = 2$, and $n = [K : \mathbb{Q}]$ if the degree of the extension is at least 3.

For a smooth projective geometrically irreducible curve C of genus g, defined over $\mathbb{F}_q$, the natural analogue of the short interval implicit in Conjecture 3.4 is the following set

Definition 3.3. Let $E = m_1\mathfrak{p}_1 + \cdots + m_s\mathfrak{p}_s$ be an effective divisor on C, and let f_0 be a regular function on the complement of E. The *interval* (*of size E around f_0*) is the set

$$\begin{aligned} I(f_0, E) &\overset{\text{def}}{=} \left\{ \begin{array}{c} \text{regular functions } h \text{ on } C\backslash\text{supp}(E) \text{ such} \\ \text{that } \nu_{\mathfrak{p}_i}(h - f_0) \geq -m_i \ \text{ for all } \ 1 \leq i \leq s \end{array} \right\} \\ &= f_0 + H^0\big(C, \mathscr{O}(E)\big), \end{aligned} \tag{19}$$

where $H^0(C, \mathscr{O}(E))$ is the space of regular functions on $C\backslash\{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ with a pole of order at most m_i at each point $\mathfrak{p}_i$, for $1 \leq i \leq s$.

The interval $I(f_0, E)$ is a *short interval* if the order of the pole of f_0 at each $\mathfrak{p}_i$ is strictly greater than m_i.

The value that serves as our prime count in any short interval $I(f_0, E)$ is

$$\pi_C(I(f_0, E)) \stackrel{\text{def}}{=} \# \left\{ \begin{array}{c} h \in I(f_0, E) \text{ such that } h \text{ generates a} \\ \text{prime ideal in the ring of regular} \\ \text{functions on } C\backslash \text{supp}\,(E) \end{array} \right\}. \tag{20}$$

In [8] we generalize Theorem 3.1 and prove an analogue of Conjecture 3.4

Theorem 3.4. [8, Theorem A.] Let C be a smooth projective geometrically irreducible curve of genus g over $\mathbb{F}_q$, and let $k > 0$ be an integer. Let E be an effective divisor on C, f_0 a regular function on $C\backslash\text{supp}\,(E)$, and $I(f_0, E)$ a short interval. Assume in addition that $E \geq 3E_0$ for some effective divisor E_0 on C with $\deg E_0 \geq 2g + 1$ and that $\deg E < k \stackrel{\text{def}}{=}$ total order of poles of f_0 on C. Then

$$\pi_C\big(I(f_0, E)\big) \;=\; \frac{\#I(f_0, E)}{k}\Big(1 + O_k(q^{-1/2})\Big). \tag{21}$$

Furthermore, the asymptotic formula (21) is uniform in E and f_0, in the sense that the rate of convergence is independent of the choice of E and f_0.

Remark 3.6. To establish Theorem 3.4, we prove a result that is stronger in two respects. First, for any partition type of the set $\{1, 2, \ldots, k\}$, we provide an asymptotic count of rational functions $h \in I(f_0, E)$ whose associated principal divisor on $C\backslash E$ has that partition type. Second, our count holds for a class of effective divisors E satisfying positivity requirements more relaxed than the ones in Theorem 3.4.

3.3. **Prime polynomial values of linear functions in short intervals.** The quantitative Goldbach conjecture and the Hardy-Littlewood (HL) n-tuple conjecture are long standing, extensively studied, unresolved conjectures in number theory. A crucial observation is that the two problems, as well as the problems on the number of primes in short intervals and on the number of primes in arithmetic progressions, have a similar structure: they are all specific cases of counting the number of prime values of linear functions in short intervals. Let us consider the general setting: Let $L_i = b_i X + a_i$, $i = 1, \ldots, n$ be distinct primitive linear functions with $a_i, b_i \in \mathbb{Z}$ (L_i's are primitive in the sense that $\gcd(a_i, b_i) = 1$) and write $\mathcal{L} = (L_1, ..., L_n)$. The corresponding prime counting function is

$$\pi_{\mathcal{L}}(I(x, \varepsilon)) \;\stackrel{\text{def}}{=}\; \#\{h \in I(x, \varepsilon) : \text{each } L_i(h) \text{ is a prime}\}. \tag{22}$$

As in the heuristic derivation of the HL conjecture from the PNT, one may expect that

Conjecture 3.7.

$$\pi_{\mathcal{L}}(I(x, \varepsilon)) \sim \mathfrak{S}(L_1, \ldots, L_n)\frac{\#I(x, \varepsilon)}{\prod_{i=1}^{n} \log(L_i(x))} \qquad \text{as} \quad x \to \infty \tag{23}$$

where $\mathfrak{S}(L_1, \ldots, L_n)$ is a positive constant, and $0 < a_i < b_i$, $b_i^\delta < x$ or $b_i < 0$, $|b_i|^{1+\delta} < a_i$ and $|b_i|x^\alpha < a_i < |b_i|x^\beta$ for $1 < \alpha < \beta$ and for all i.

Note that if $L_1(x) = x$ and $L_2(x) = a - x$, then (23) would imply a quantitative Goldbach conjecture (for all sufficiently large even $a \in \mathbb{Z}$) and if $L_i(X) = x + a_i$, we retrieve the HL n-tuple conjecture.

In the function field setting, the Polynomial Goldbach problem and the HL problem were both settled by Pollack and Bender in [13] and independently by Bary-Soroker in [10], all in the large q limit.

In [5], we prove the function field analogue of the generalized conjecture (23) in the large q limit. For a primitive linear function $L(X) = f(t) + g(t) \cdot X$ with $f, g \in \mathbb{F}_q[t]$ and $g \neq 0$, the *height of* L is $\text{height}(L) \stackrel{\text{def}}{=} \max\{\deg(f), \deg(g)\}$. By abuse of notation, we write $\pi_{q,\mathfrak{L}}(I(f_0, \varepsilon))$ as the analogue of the counting function (22).

Theorem 3.5. [5, Theorem 1.1] Let $0 < B$ and $0 < \varepsilon < 1$ be fixed real numbers. Then the asymptotic formula

$$\pi_{q,\mathfrak{L}}(I(f_0, \varepsilon)) = \frac{\#I(f_0, \varepsilon)}{\prod_{i=1}^{n} \deg(L_i(f_0))}\Big(1 + O_B(q^{-1/2})\Big)$$

holds uniformly for all odd prime powers q, $1 \leq n \leq B$, distinct primitive linear functions $L_1(X), \ldots, L_n(X)$ defined over $\mathbb{F}_q[t]$ each of height at most B, and monic $f_0 \in \mathbb{F}_q[t]$ of degree in the interval $\frac{2}{\varepsilon} \leq \deg f_0 \leq B$.

3.4. **Correlation between primes on curves.** As in the Section 3.2, one may ask similar question about the correlation of primes in the setting of smooth projective curves over a finite field. For a smooth projective geometrically irreducible curve C over a finite field $\mathbb{F}_q$, we let $E = m_1\mathfrak{p}_1 + \cdots + m_s\mathfrak{p}_s$ be an effective divisor on C. Let $\sigma \stackrel{\text{def}}{=} (\sigma_1, ..., \sigma_n)$ be an n-tuple of rational functions σ_i on C, regular on $C\backslash\text{supp}\,(E)$. The corresponding prime counting function on a short interval $I(f_0, E)$ (as in Definition 3.3) is now

$$(24)\qquad \pi_{C,\sigma}(I(f_0, E)) \;\stackrel{\text{def}}{=}\; \#\left\{\begin{matrix} h \in I(f_0, E) \text{ such that } h + \sigma_1, ..., h + \sigma_n \text{ generate prime} \\ \text{ideals in the ring of regular functions on } C\backslash\text{supp}\,(E) \end{matrix}\right\}.$$

In [4], we prove the following analogue of Theorem 3.5, for $f_0, \sigma_1, ..., \sigma_n$ distinct regular functions on $C\backslash\text{supp}\,(E)$ satisfying $-\nu_{\mathfrak{p}}(f_0) > m_{\mathfrak{p}}$ and $\nu_{\mathfrak{p}}(f_0) \neq \nu_{\mathfrak{p}}(\sigma_i)$ for each $1 \leq i \leq n$.

Theorem 3.6. Fix an integer $B > n$. If $\text{char}\,\mathbb{F}_q \neq 2$ and $E \geq 3E_0$ for some effective divisor E_0 on C with $\deg E_0 \geq 2g + 1$, then the asymptotic formula

$$\pi_{C,\sigma}\big(I(f_0, E)\big) \;=\; \frac{\#I(f_0, E)}{\prod_{i=1}^{n} \deg \text{div}(f_0 + \sigma_i)|_{C\backslash E}}\Big(1 + O_B\big(q^{-1/2}\big)\Big)$$

holds uniformly for all E and $f_0, \sigma_1, \ldots, \sigma_n$ as above satisfying $\deg\big(\text{div}(f_0 + \sigma_i)\big|_E\big) < B$, and as $q \to \infty$ an odd prime power.

3.5. **Sum of squares in short intervals.** In this project we prove an asymptotic density for the number of polynomials over a finite field that can be represented as a sum of two squares.

We say that an integer n is *representable as sum of two squares*, in short *representable*, if there exist $a, b \in \mathbb{Z}$ such that $n = a^2 + b^2$. One may observe that an integer is representable if and only if it is a norm of an element in the Gaussian integers $\mathbb{Z}[i]$. A famous theorem of Landau [36], asserts that the asymptotic density of representable integers n in the interval $I = [1, x]$ is $\frac{K}{\sqrt{\log x}}$, where K is the Landau-Ramanujan constant. More precisely, let

$$(25)\qquad b(n) \stackrel{\text{def}}{=} \begin{cases} 1, & n \text{ is representable} \\ 0, & \text{otherwise.} \end{cases}, \qquad \langle b(n)\rangle_{n\in I} \stackrel{\text{def}}{=} \frac{1}{|I|}\sum_{n\in I} b(n),$$

here $b(n)$ is the characteristic function of representable integers and $\langle b(n)\rangle_{n\in I}$ is its mean value in the interval I.

Theorem 3.8 (Landau)**.**

$$\langle b(n)\rangle_{n\in I} \sim K\frac{1}{\sqrt{\log x}}, \qquad x\to\infty. \tag{26}$$

In view of Theorem 3.8 one would expect that in an interval $I = (x-\Phi(x), x-\Phi(x)]$ the same asymptotic density should hold, i.e., the mean value $\langle b(n)\rangle_{n\in I}$ is approximately $K\frac{1}{\sqrt{\log x}}$ as $x\to\infty$ and for $\Phi(x) > \sqrt{\log x}$. The problem of estimating the mean value of $b(n)$ in such intervals has a long history. When restricting to all x outside a set of asymptotic density 0, the correct upper and lower bounds are known; See Friedlander [20, 19] and Hooley [27] for upper bounds; Plaskin [43], Harman [23], and Hooley [27] for lower bounds. However, for $\Phi(x) = (\log x)^A$, $A > \frac{1}{2}$ Balog and Wooley [3] show a Maier type phenomenon, which implies that the expected density fails. Thus, a natural restriction is considering $\Phi(x) = x^\epsilon$ with fixed $0<\epsilon<1$. It is a folklore conjecture that

Conjecture 3.9. For any fixed $0<\epsilon<1$, let $I(x,\varepsilon) = (x-x^\varepsilon, x+x^\varepsilon]$ be a short interval. Then (26) holds.

Using methods of Ingham, Montgomery and Huxley for primes in short intervals, one can confirm Conjecture 3.9 for $\epsilon > \frac{7}{12}$ unconditionally and for $\epsilon > \frac{1}{2}$ assuming the Riemann Hypothesis, see [26].

In [6], we resolve a function field analogue of the above conjecture in the limit of a large finite field. Consider the ring $\mathbb{F}_q[\sqrt{-t}]$ as the analogue of $\mathbb{Z}[i]$. A monic polynomial $f\in\mathbb{F}_q[t]$ is *representable* if it is a norm of an element from $\mathbb{F}_q[\sqrt{-t}]$. That is, a monic polynomial $f\in\mathbb{F}_q[t]$ is representable if there exist polynomials $A, B\in\mathbb{F}_q[t]$ such that $f = A^2+tB^2$. For a monic polynomial $f\in\mathbb{F}_q[t]$ of degree k we define $b_q(f)$ and $\langle b_q(f)\rangle_{f\in I}$ to be the analogues of (25). Bary-Soroker, Smilansky and Wolf in [11] show an analogue of Theorem 3.8

$$\langle b_q(f)\rangle_{f\in I(t^k,k-1)} = \frac{1}{4^k}\binom{2k}{k} + O_k(q^{-1}) \tag{27}$$

where the implied constant depends only on k.

Our main result in this work is a function field analogue of Conjecture 3.9, namely of Landau's theorem for short intervals

Theorem 3.10. For odd q, $k>2$, $1>\varepsilon\geq\frac{2}{k}$, and $f_0\in\mathbb{F}_q[t]$ monic of degree k, we have

$$\langle b_q(f)\rangle_{I(f_0,\epsilon)} = \frac{1}{4^k}\binom{2k}{k} + O_k(q^{-1/2}) \tag{28}$$

where the implied constant depends only on k. For $0<\epsilon<\frac{2}{k}$, (28) no longer holds.

3.6. **Diophantine approximations in function fields.** A main topic in Diophantine approximation deals with the inhomogeneous approximation of a real number. In this project we prove a function field analogue of this problem.

Function fields analogues of Diophantine approximation have been studied since the works of Artin [1] and Mahler [39]. Recently, this subject has regained interest, parallel to a significant progress in the real case, see for example [32, 33, 48]. In [9], we prove Hausdorff dimension results for exceptional sets connected to inhomogeneous Diophantine approximation and determine explicitly Cassels' constant, both in the function fields setting. In the (classical) real case, similar Hausdorff dimension results have also been proved. However, only estimates have been obtained for Cassels' constant. Our methods of proof use connections to linear algebra and tree-like collections for bounding the Hausdorff dimension and are much inspired by a paper of Davenport and Lewis [17].

For a real number θ, denote $\langle\theta\rangle \overset{\text{def}}{=} \theta - \left\lfloor \theta + \frac{1}{2} \right\rfloor$ and let $|\theta|$ denote the absolute value of θ. In these notation, $|\langle\theta\rangle|$ is the distance from θ to the integers. Given two real numbers θ and γ define the *approximation constant* of θ with respect to γ and the *badly approximating set* of θ as

$$c(\theta,\gamma) \overset{\text{def}}{=} \inf_{n\neq 0} |n| \cdot |\langle n\theta - \gamma\rangle| \qquad BA_\theta \overset{\text{def}}{=} \left\{\gamma \in \mathbb{R} \ : \ c(\theta,\gamma) > 0\right\}. \tag{29}$$

We summarize two known results

Theorem 3.7. The set BA_θ satisfies

(1) [Berend and William, [14]] For every $\theta \in \mathbb{R} \setminus \mathbb{Q}$, the set BA_θ has zero Lebesgue measure.
(2) [Tseng, [50]] For every $\theta \in \mathbb{R}$, the set BA_θ has Hausdorff dimension 1.

In particular, Theorem 3.7(2) states that BA_θ is not empty. This leads to the definitions

$$c(\theta) \overset{\text{def}}{=} \sup_{\gamma} c(\theta,\gamma) \qquad c \overset{\text{def}}{=} \inf_{\theta} c(\theta). \tag{30}$$

The constant c is known as the *Cassels' constant*. Khinchine [31] proved that $c > 0$, and the problem of finding the exact value of c was posed by Cassels [16, p.86]. According to [41], the best known estimates of c are $\frac{3}{32} \leq c \leq \frac{68}{483}$. We define the natural function fields analogues of (29) and (30). For $\theta,\ \gamma \in \mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)$ and $N \in \mathbb{F}_q[t]$ a non-zero polynomial, let

$$c(\theta,\gamma) \overset{\text{def}}{=} \inf_{0\neq N} |N| \cdot |\langle N\theta - \gamma\rangle| \quad c(\theta) \overset{\text{def}}{=} \sup_{\gamma} c(\theta,\gamma) \quad c \overset{\text{def}}{=} \inf_{\theta} c(\theta) \tag{31}$$

$$BA_\theta \overset{\text{def}}{=} \left\{\gamma \in \mathbb{F}_q\left(\left(\frac{1}{t}\right)\right) \ : \ c(\theta,\gamma) > 0\right\}. \tag{32}$$

We prove an analogue of Theorem 3.7(2)

Theorem 3.8. [9, Theorem 3.5] For every $\theta \in \mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)$, $\dim\left(BA_\theta\right) = 1$.

We determine the Cassels' constant in the function field setting

Theorem 3.9. [9, Theorem 3.9] Cassels' constant is $c = q^{-2}$.

Remark 3.11. We remark that in [9] we prove more than Theorems 3.8 and 3.9. In fact, we prove those theorems in higher dimensions, namely for $(\mathbb{F}_q((\frac{1}{t})))^d$ and for general weights.

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